

Commutative Quantum Operator Algebras

Bong H. Lian and Gregg J. Zuckerman

ABSTRACT. A key notion bridging the gap between *quantum operator algebras* [22] and *vertex operator algebras* [4][8] is the definition of the commutativity of a pair of quantum operators (see section 2 below). This is not commutativity in any ordinary sense, but it is clearly the correct generalization to the quantum context. The main purpose of the current paper is to begin laying the foundations for a complete mathematical theory of *commutative quantum operator algebras*. We give proofs of most of the relevant results announced in [22], and we carry out some calculations with sufficient detail to enable the interested reader to become proficient with the algebra of commuting quantum operators.

We dedicate this paper to the memory of Feza Gürsey.

1 Introduction

Formal infinite sums of linear operators in a vector space have been explicitly appearing in quantum field theory since the late 1920's. A particular class of such formal sums is studied by the authors [22] in the context of conformal field theory and string theory. Let V be a \mathbf{Z} doubly graded vector space $V = \oplus V^n[m]$. Let z be a formal variable (later to be thought of as a point in the punctured complex plane). A homogeneous quantum operator is a formal power series $u(z) = \sum u(n)z^{-n-1}$ of degrees, say d_1, d_2 , where the coefficients $u(n)$ are linear maps in V with degrees $d_1, d_2 - n - 1$. A quantum operator is a finite sum of homogeneous quantum operators, and we denote the space of all quantum operators in V by $QO(V)$.

The famous vertex operator construction of the old dual resonance model provides the first concrete and nontrivial examples of quantum operators to appear in the physics literature, and later in the mathematics literature. Here the space V is the Fock space, which is only singly graded by weight. The famous construction of Virasoro yields a quantum operator $L(z)$ acting in V and having the property that the coefficients $L(n)$ of $L(z)$ span, together with the identity operator, a central extension of the Lie algebra of Fourier polynomial vector fields on the circle.

The BRST quantization of classical strings provides examples of quantum operators with nonzero ghost number. Now the space V is a tensor product of a so-called ghost Fock space

⁰1991 Mathematics Subject Classification. Primary 81T70, 17B68.

⁰B.H.L. is supported by grant DE-FG02-88-ER-25065. G.J.Z. is supported by NSF Grant DMS-9307086 and DOE Grant DE-FG02-92-ER-25121.

with the state space of a conformal field theory. A special quantum operator known as the BRST current $J(z)$ plays a key role: the coefficient $J(0)$ becomes, under suitable conditions, a cohomology operator in the space V . The much studied BRST cohomology groups [17][11][6][7][23][24][25] are then defined to be the cohomology groups associated to the differential, $J(0)$. Next to $J(z)$ the most important quantum operator in the theory is the ghost field $b(z)$, whose coefficient $b(1)$ induces in the cohomology the structure of a Batalin-Vilkovisky algebra. Much of the recent interest in BRST string theory has centered on the myriad algebraic structures possessed both by BRST cohomology as well as the complex V itself.

Both the dual resonance theory and the BRST-quantized string theory deal with the above-mentioned quantum operators as distinguished members of certain infinite dimensional linear systems of quantum operators. The theory of vertex operator algebras (VOAs) [4][8] establishes a remarkable but rather complex foundation for the study of such systems. In [26], the authors successfully adapted the theory of VOAs to the theory of BRST cohomology. In an attempt to better understand this application of VOA theory, the authors recently introduced a more general class of systems called quantum operator algebras (QOAs) [22]. The latter can also be viewed as elementary generalizations of the operator algebras well-known in linear algebra and mathematical physics. We want to emphasize that the operators, rather than the states, are fundamental in our new point of view.

A key notion bridging the gap between QOAs and VOAs is the definition of commutativity of a pair of quantum operators [22] (see also [8][9][5][21]). This is not commutativity in any ordinary sense, but it is clearly the correct generalization to the quantum context. The main purpose of the current paper is to begin laying the foundations for a complete mathematical theory of commutative quantum operator algebras. We give proofs of most of the relevant results announced in [22], and we carry out some calculations with sufficient detail to enable the interested reader to become proficient with the algebra of commuting quantum operators.

The main features of this operator calculus were discovered by the quantum physicists. These features are certainly captured by the theory of VOAs, and we have benefitted enormously from the insights in the VOA literature, especially with regard to the notion of commutativity. However, because VOA theory emphasizes states in V over quantum operators in $QO(V)$, we feel that much of the beauty and simplicity of the physicists' operator calculus gets lost in the translation. Moreover, we have found that the construction of the main nontrivial examples of VOAs becomes much simpler and much closer to the original physical inspiration when carried out in the language of commutative quantum operator algebras. Thus, a subsidiary purpose of this paper is to present in fairly great detail the construction of the simplest CQOAs that enter into the BRST construction.

Here is a brief summary of the contents of this paper:

In section 2, we state the definitions of our main concepts: quantum operators, matrix elements, Wick products, iterated Wick products, the infinitely many "circle" products, the operator product expansion, the notions of locality and commutativity, and finally the corresponding notions of local and commutative quantum operator algebras. We discuss some elementary

known facts about commutativity. We then define the notion of a semi-infinite commutative algebra, abstracted from our theory of CQOAs.

In section 3, we introduce what we call the Wick calculus, which deals with operator products of the form $:t(z)u(z):v(w)$ as well as $t(z):u(w)v(w):$ under the assumption that the quantum operators $t(z)$, $u(z)$, and $v(z)$ are *pairwise commutative*. Here, $:t(z)u(z):$ denotes the Wick or normal ordered product of $t(z)$ with $u(z)$. The Wick calculus is essential for both computations as well as for theoretical issues, such as the explicit construction of CQOAs. Section 3 continues with the construction of the CQOA $O(b, c)$, which acts in the ghost Fock space of the BRST construction. This section concludes with a construction of the CQOA $O_\kappa(L)$, which arose originally in the seminal work of BPZ [3], and which acts in the state space of any conformal field theory having central charge κ . Both examples are special, in that the two algebras are spanned by iterated Wick products of derivatives of finitely many generating quantum operators. In fact, we exhibit explicit bases consisting of such iterated products.

In section 4, we discuss the BRST construction in the language of what we call conformal QOAs. Given a conformal QOA O with central charge κ , we form the tensor product $C^*(O) = O(b, c) \otimes O$. We then construct the special quantum operator $J(z)$, referred to above as the BRST current. We also give a simple characterization of $J(z)$. We then recall the famous result that the coefficient $J(0)$ is square-zero if and only if $\kappa = 26$. After that we specialize to this central charge. As in our past paper [22], there is a change in point of view: rather than discussing $J(0)$ as a differential in the BRST state space we instead consider $[J(0), -]$ as a square-zero derivation of the BRST quantum operator algebra, $C^*(O)$. Thus, we view BRST quantization in the operator picture, which is closer to the original physical context.

A particular feature of section 4 is our calculation for arbitrary central charge κ of the graded commutator $[J(0), J(0)] = 2J(0)^2$, which can be obtained from the calculation of the operator product, $J(z)J(w)$. Appealing to the abstract development in section 3, we carry out the evaluation of this special operator product via the Wick calculus. This computation provides a model for the kinds of calculations made routinely by conformal field theorists since the pioneering work of BPZ [3].

The main result of section 4 is Theorem 4.5, which states that the Wick product induces a graded commutative associative product on the cohomology of $C^*(O)$ with respect to the derivation, $[J(0), -]$. This theorem first appeared in work of E. Witten [32], who called the ghost number zero subalgebra the “ground ring of a string background”. An approach to this theorem via VOA theory appears in [26]. The approach in the current paper is via CQOA theory, and hopefully appears as a significant simplification of our earlier work, as well as a clear exposition of Witten’s work.

In section 5, we develop the theory of the ghost field, $b(z)$, and its coefficient, $b(1)$. As a preparation, we remind the reader of the definition of a Batalin-Vilkovisky (BV) operator and BV algebra. The main result of section 5 is Theorem 5.2, which states that the operator $b(1)$ induces a BV operator acting in the BRST cohomology algebra. Thus the cohomology becomes a BV algebra. This theorem was inspired by work of Witten and Zwiebach [34], and

first appeared in [26], where it was derived via identities from VOA theory. Again, we hope that our new approach via CQOA theory sheds light on both our original discovery as well as the physical inspiration.

Nontrivial examples of BV structure in BRST cohomology appear in section 3 of [26] as well as section 6 of [22]. We plan to return to these explicit examples in future work. An application of these BV structures can be found in [27]. The notion of the BRST cohomology of the so-called W -algebras has recently been investigated by Bouwknegt, McCarthy and Pilch who again find a BV algebra structure (see [2] and references therein). The W -algebras can be viewed as a generalization of the Virasoro algebra.

We note here that the abstract notions of a BV operator and BV algebra were discovered quite independently of string theory by the mathematician J. L. Koszul [20] (see also [19]). At roughly the same time Batalin and Vilkovisky applied a particular BV operator to their “antifield” approach to Lagrangian quantum field theory. In a future work, we plan to review the relationships between our own work and the relevant papers of Koszul, Batalin and Vilkovisky, E. Getzler, A. Schwarz, and others. We hope to highlight the many intriguing parallel developments in physics and mathematics. Such parallels seem to be the hallmark of string theory itself.

Acknowledgments: We thank G. Moore for many informative discussions about the operator product expansions. We thank Y. Kosmann-Schwarzbach for sending us her recent report, and F. Akman for carefully proofreading our manuscript.

2 Quantum Operator Algebras

Let V be a *bounded* graded vector space. By this, we mean V is a \mathbf{Z} doubly graded vector space $V = \oplus V^n[m]$ such that for each fixed n , $V^n[m] = 0$ for all but finitely many negative m ’s. The degrees of a homogeneous element v in $V^n[m]$ will be denoted by $|v| = n$, $||v|| = m$ respectively. In physical applications, $|v|$ will be the fermion or ghost number of v . In conformal field theory, $||v||$ will be the conformal dimension or weight of v .

Let z be a formal variable with degrees $|z| = 0$, $||z|| = -1$. Then it makes sense to speak of a *homogeneous* (biinfinite) formal power series

$$u(z) = \sum_{n \in \mathbf{Z}} u(n) z^{-n-1} \quad (2.1)$$

of degrees $|u(z)|$, $||u(z)||$ where the coefficients $u(n)$ are homogeneous linear maps in V with degrees $|u(n)| = |u(z)|$, $||u(n)|| = -n - 1 + ||u(z)||$. Note then that the terms $u(n) z^{-n-1}$ indeed have the same degrees $|u(z)|$, $||u(z)||$ for all n . We call a finite sum of such homogeneous series $u(z)$ a *quantum operator* on V , and we denote the linear space of quantum operators as $QO(V)$.

Notations: By the expression $(z - w)^n$, n an integer, we usually mean its formal power series expansion in the region $|z| > |w|$. Thus $(z - w)^{-2}$ and $(-w + z)^{-2}$ are different, as power series. When such expressions are to be regarded as rational functions rather than formal series, we will explicitly mention so. When $A(z) = \sum A(n)z^{-n-1}$ is a formal series with coefficients $A(n)$ in whatever linear space, we define $\text{Res}_z A(z) = A(0)$, $A(z)^+ = \sum_{n \geq 0} A(n)z^{-n-1}$, $A(z)^- = \sum_{n < 0} A(n)z^{-n-1}$, $\partial A(z) = \sum -(n+1)A(n)z^{-n-2}$. If $u(z), u'(z)$ belong to QOAs O, O' respectively, we abbreviate $u(z) \otimes u'(z)$, as an element of $O \otimes O'$, simply as $u(z)u'(z)$. When no ambiguity occurs, we denote $|u(z)|, ||u(z)||$ simply as $|u|, ||u||$. The restricted dual of a graded vector space V is denoted $V^\#$. If $A_1(z), A_2(z), \dots$ are quantum operators, an arbitrary matrix element $\langle x, A_1(z_1)A_2(z_2) \cdots y \rangle$ with $x \in V^\#, y \in V$, is denoted as $\langle A_1(z_1)A_2(z_2) \cdots \rangle$. In the interest of clarity, we often write signs like $(-1)^{|t||u|}$ simply as \pm . This convention is used only when the sign arises from permutation of elements. When in doubt, the reader can easily recover the correct sign from such a permutation.

Given two quantum operators $u(z), v(z)$, we write

$$: u(z)v(w) := u(z)^-v(w) + (-1)^{|u||v|}v(w)u(z)^+. \quad (2.2)$$

Because V is bounded, it's easy to check that if we replace w by z , the right hand side makes sense as a quantum operator and hence defines a nonassociative product on $QO(V)$. It is called the *Wick product*. Similarly given $u_1(z), \dots, u_n(z)$, we define $: u_1(z_1) \cdots u_n(z_n) :$ inductively as $: u_1(z_1)(: u_2(z_2) \cdots u_n(z_n) :) :$.

Definition 2.1 For each integer n we define a product on $QO(V)$:

$$u(w) \circ_n v(w) = \text{Res}_z u(z)v(w)(z - w)^n - (-1)^{|u||v|} \text{Res}_z v(w)u(z)(-w + z)^n. \quad (2.3)$$

Explicitly we have:

$$u(z) \circ_n v(z) = \begin{cases} \frac{1}{(-n-1)!} : \partial^{-n-1} u(z) v(z) : & \text{if } n < 0 \\ [(\sum_{m=0}^n \binom{n}{m} u(m)(-z)^{n-m}), v(z)] & \text{if } n \geq 0. \end{cases} \quad (2.4)$$

If A is a homogeneous linear operator on V , then it's clear that the graded commutator $[A, -]$ is a graded derivation of each of the products \circ_n . Since $u(z) \circ_0 v(z) = [u(0), v(z)]$, we have

Proposition 2.2 For any $t(z), u(z), v(z)$ in $QO(V)$ and n integer, we have

$$t(z) \circ_0 (u(z) \circ_n v(z)) = [t(z) \circ_0 u(z)] \circ_n v(z) \pm u(z) \circ_n [t(z) \circ_0 v(z)],$$

ie. $t(z) \circ_0$ is a derivation of every product in $QO(V)$.

Proposition 2.3 For $u(z), v(z)$ in $QO(V)$, the following equality of formal power series in two variables holds:

$$u(z)v(w) = \sum_{n \geq 0} u(w) \circ_n v(w)(z-w)^{-n-1} + :u(z)v(w):. \quad (2.5)$$

Proof: We have $u(z)v(w) = [u(z)^+, v(w)] + :u(z)v(w):$. On the other hand by inverting the second eqn. in (2.4), we get

$$[u(m), v(w)] = \sum_{n=0}^m \binom{m}{n} u(w) \circ_n v(w) w^{m-n}. \quad (2.6)$$

Thus we have

$$\begin{aligned} [u(z)^+, v(w)] &= \sum_{m \geq n \geq 0} \binom{m}{n} u(w) \circ_n v(w) w^{m-n} z^{-m-1} \\ &= \sum_{n \geq 0} u(w) \circ_n v(w) \frac{1}{n!} \partial_w^n (z-w)^{-1} \\ &= \sum_{n \geq 0} u(w) \circ_n v(w) (z-w)^{-n-1}. \quad \square \end{aligned} \quad (2.7)$$

In the sense of the above Proposition, $:u(z)v(w):$ is the *nonsingular part* of the *operator product expansion* (2.5), while $u(w) \circ_n v(w)(z-w)^{-n-1}$ is the *polar part* of order $-n-1$ (see [3]). The products \circ_n will become important for describing the algebraic and analytic structures of certain algebras of quantum operators. Thus we introduce the following mathematical definitions:

Definition 2.4 A graded subspace \mathcal{A} of $QO(V)$ containing the identity operator and closed with respect to all the products \circ_n is called a *quantum operator algebra*. We say that $u(z)$ is *local* to $v(z)$ if $u(z) \circ_n v(z) = 0$ for all but finitely many positive n . A QOA \mathcal{A} is called *local* if its elements are pairwise mutually local.

We observe that for any element $a(z)$ of a QOA, we have $a(z) \circ_{-2} 1 = \partial a(z)$. Thus a QOA is closed with respect to formal differentiation.

Proposition 2.5 Let $u(z), v(z)$ be quantum operators, and N a nonnegative integer. If $u(z) \circ_n v(z) = 0$ for $n \geq N$, then $\langle u(z)v(w) \rangle$ represents a rational function in $|z| > |w|$ with poles along $z = w$ of order at most N .

Proof: By eqn (2.5), we have

$$\langle u(z)v(w) \rangle = \sum_{n \geq 0} \langle u(w) \circ_n v(w) \rangle (z-w)^{-n-1} + \langle : u(z)v(w) : \rangle. \quad (2.8)$$

It is trivial to check that $\langle : u(z)v(w) : \rangle, \langle u(w) \circ_n v(w) \rangle \in \mathbf{C} [z^{\pm 1}, w^{\pm 1}]$. Thus our claim follows immediately. \square

Lemma 2.6 *Let $u(z)$ be local to $v(z)$, and $\langle u(z)v(w) \rangle$ represent the rational function $f(z, w)$. Then for $|w| > |z - w|$,*

$$f(z, w) = \sum_{n \in \mathbf{Z}} \langle u(w) \circ_n v(w) \rangle (z-w)^{-n-1}. \quad (2.9)$$

Proof: The Laurent polynomial $\langle : u(z)v(w) : \rangle$ in the above region is just $\sum_{i \geq 0} \frac{1}{i!} \langle : (\partial^i u(w))v(w) : \rangle (z-w)^i$. Now apply eqn. (2.4). \square

We note that none of the products \circ_n is associative in general. However it clearly makes sense to speak of the left, right or two sided ideals in a QOA as well as homomorphisms of QOAs and they are defined in an obvious way. For example, a linear map $f : O \rightarrow O'$ is a homomorphism if $f(u(z) \circ_n v(z)) = fu(z) \circ_n fv(z)$ for all $u(z), v(z) \in O$, and $f(1) = 1$.

Definition 2.7 *Two quantum operators $u(z), v(z)$ are said to commute if they are mutually local, and $\langle u(z)v(w) \rangle, \pm \langle v(w)u(z) \rangle$ represent the same rational function. This is equivalent (Proposition 2.5) to the following: for some $N \geq 0$, $(z-w)^N \langle u(z)v(w) \rangle = \pm (z-w)^N \langle v(w)u(z) \rangle$ as Laurent polynomials. We call a QOA O whose elements pairwise commute a commutative QOA.*

Proposition 2.8 *If $u(z), v(z)$ commute, then for all m*

$$[u(m), v(w)] = \sum_{n \geq 0} \binom{m}{n} u(w) \circ_n v(w) w^{m-n}. \quad (2.10)$$

Proof: The case $m \geq 0$ is obtained by inverting the second eqn. in (2.4). Since $u(z)v(w) = [u(z)^+, v(w)] + : u(z)v(w) :$ and $v(w)u(z) = \mp [u(z)^-, v(w)] \pm : u(z)v(w) :$, it follows from commutativity that $\langle [u(z)^-, v(w)] \rangle$ represents the same rational function as $-\langle [u(z)^+, v(w)] \rangle$ does, which is just $-\sum_{n \geq 0} \frac{u(w) \circ_n v(w)}{(z-w)^{n+1}}$. This gives

$$[u(z)^-, v(w)] = - \sum_{n \geq 0} u(w) \circ_n v(w) (-w+z)^{-n-1}. \quad (2.11)$$

Taking $Res_z[u(z)^-, v(w)]z^m$ for $m < 0$ gives the desired result. \square

The notion of commutativity here is closely related to the physicists' notion of duality in conformal field theory[28]. Frenkel-Lepowsky-Meurman have reformulated the axioms of a VOA in terms of what they call rationality, associativity and commutativity. The notion of commutativity in Definition 2.7 is essentially the same as FLM's. This notion has also been reformulated in the language of formal variables in [5].

Definition 2.9 Let \mathcal{O} be a bounded graded space equipped with a distinguished vector 1 and a set of bilinear products \circ_n , with $|1| = ||1|| = 0$, $|\circ_n| = 0$, $||\circ_n|| = -n - 1$. We call \mathcal{O} a semi-infinite commutative algebra if the following holds: for homogeneous $u, v \in \mathcal{O}$,

- (i) $u \circ_n 1 = \delta_{n,-1}u$ for $n \geq -1$;
- (ii) $u \circ_n v = (-1)^{|u||v|} \sum_{p \in \mathbf{Z}} (-1)^{p+1} (v \circ_p u) \circ_{n-p-1} 1$.

Note that the sum in (ii) is finite because by (i), the summand is zero for $p < n$, and by boundedness of the space \mathcal{O} , $v \circ_p u = 0$ for $p \gg 0$. Note that the leading term on the RHS of (ii) is $\pm v \circ_n u$. Thus the products \circ_n are graded commutative up to "higher order corrections". We claim that in a $\frac{\infty}{2}$ -commutative algebra, $1 \circ_n t = \delta_{n,-1}t$ for all n . Applying (i) to (ii) (with $u = 1, v = t$), we have for $n \geq -1$,

$$1 \circ_n t = \sum (-1)^{p+1} (t \circ_p 1) \circ_{n-p-1} 1 = (t \circ_{-1} 1) \circ_n 1 = \delta_{n,-1}t. \quad (2.12)$$

Now let $u = t, v = 1$. Applying (2.12), (ii) becomes for $n \leq -2$,

$$\begin{aligned} t \circ_n 1 &= \sum (-1)^{p+1} (1 \circ_p t) \circ_{n-p-1} 1 \\ &= t \circ_n 1 + \sum_{p \leq -2} (-1)^{p+1} (1 \circ_p t) \circ_{n-p-1} 1. \end{aligned} \quad (2.13)$$

Thus $\sum_{p \leq -2} (-1)^{p+1} (1 \circ_p t) \circ_{n-p-1} 1 = 0$. For $n = -2$, this reads $0 = -(1 \circ_{-2} t) \circ_{-1} 1 = -1 \circ_{-2} t$. By induction on n , we have $1 \circ_n t = 0$ for $n \leq -2$.

Proposition 2.10 Let O be a local QOA. Then O is commutative iff it is a $\frac{\infty}{2}$ -commutative algebra.

Proof: The unital property (i) follows from eqn. (2.4): $u(z) \circ_n 1 = \delta_{n,-1}u(z)$ for $n \geq -1$. Let f, g be the rational functions represented by $\langle u(z)v(w) \rangle$ and $\langle v(w)u(z) \rangle$ respectively. By Lemma 2.6, we have

$$\begin{aligned} f(z, w) &= \sum_{n \in \mathbf{Z}} \langle u(w) \circ_n v(w) \rangle (z - w)^{-n-1} \quad \text{for } |w| > |z - w| \\ g(z, w) &= \sum_{p \in \mathbf{Z}} \langle v(z) \circ_p u(z) \rangle (w - z)^{-p-1} \quad \text{for } |z| > |z - w| \end{aligned}$$

$$\begin{aligned}
&= \sum_{p \in \mathbf{Z}, m \geq 0} (-1)^{p+1} \frac{1}{m!} \langle \partial^m (v(w) \circ_p u(w)) \rangle (z-w)^{m-p-1} \quad \text{for } |z|, |w| > |z-w| \\
&= \sum_{p, m \in \mathbf{Z}} (-1)^{p+1} \langle (v(w) \circ_p u(w)) \circ_{-m-1} 1 \rangle (z-w)^{m-p-1} \\
&= \sum_{p, n \in \mathbf{Z}} (-1)^{p+1} \langle (v(w) \circ_p u(w)) \circ_{n-p-1} 1 \rangle (z-w)^{-n-1}.
\end{aligned} \tag{2.14}$$

It's now clear that $f(z, w) = \pm g(z, w)$ holds iff the identity (ii) holds. \square

3 Wick's calculus

In this section, we derive a number of useful formulas relating various iterated products among three quantum operators. Most of these formulas are well-known to physicists who are familiar with the calculus of operator product expansions. We will also include a lemma on commutativity.

Let $t(z), u(z), v(z)$ be homogeneous quantum operators which pairwise commute.

Lemma 3.1 (see [21]) *For all n , $t(z) \circ_n u(z)$ and $v(z)$ commute.*

Proof: We include Li's proof here for completeness. For a positive integer N , $(z-w)^{2N}$ is a binomial sum of terms $(z-x)^i (x-w)^{2N-i}$, $i = 1, \dots, 2N$. So $(z-w)^{N+2N} (t(z) \circ_n u(z)) v(w)$ is a binomial sum of terms

$$Res_x \left((z-w)^N (z-x)^i (x-w)^{2N-i} (t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n) v(w) \right). \tag{3.15}$$

We want to show that for large enough N , and for $0 \leq i \leq 2N$, term by term we have

$$\begin{aligned}
&(z-w)^N (z-x)^i (x-w)^{2N-i} (t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n) v(w) \\
&= \pm (z-w)^N (z-x)^i (x-w)^{2N-i} v(w) (t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n).
\end{aligned} \tag{3.16}$$

Consider two cases: $i \geq N$ and $i < N$. By assumption, $(z-x)^k (t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n) = 0$ for all large enough k . So for large enough N , (3.16) holds for $i \geq N$. Similarly for $i < N$, $(z-w)^N (x-w)^{2N-i} (t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n) v(w)$ coincides with $\pm (z-w)^N (x-w)^{2N-i} v(w) (t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n)$. This shows that (3.16) holds for each i . \square

This lemma is useful for showing existence of commutative QOAs: it says that given a set of pairwise commuting quantum operators, the QOA generated by the set is commutative. We now develop some abstract tools for studying the structure of commutative QOAs.

Applying (2.5), we have

$$\begin{aligned}
& : t(z)u(z) : v(w) \\
&= (t(z)^- u(z) \pm u(z)t(z)^+) v(w) \\
&= t(z)^- u(z)v(w) \pm u(z)v(w)t(z)^+ \pm u(z)[t(z)^+, v(w)] \\
&= \sum_{n \geq 0} : t(z)(u(w) \circ_n v(w)) : (z-w)^{-n-1} + : t(z)u(z)v(w) : \\
&\quad \pm \sum_{n, m \geq 0} u(w) \circ_m (t(w) \circ_n v(w))(z-w)^{-n-m-2} + \\
&\quad \pm \sum_{n \geq 0} : u(z)(t(w) \circ_n v(w)) : (z-w)^{-n-1}.
\end{aligned} \tag{3.17}$$

Similarly,

$$\begin{aligned}
& t(z) : u(w)v(w) : \\
&= \pm [u(w)^-, t(z)]v(w) \pm u(w)^- t(z)v(w) \pm t(z)v(w)u(w)^+ \\
&= \pm \sum_{n, m \geq 0} (-1)^{n+1} u(w) \circ_n (t(w) \circ_m v(w))(z-w)^{-n-m-2} \\
&\quad \pm \sum_{n \geq 0} (-1)^{n+1} : u(z) \circ_n t(z) v(w) : (z-w)^{-n-1} \\
&\quad \pm \sum_{n \geq 0} : u(w) t(w) \circ_n v(w) : (z-w)^{-n-1} \pm : u(w)t(z)v(w) :
\end{aligned} \tag{3.18}$$

Lemma 3.2 *The following equalities hold in $|w| > |z - w|$:*

$$\begin{aligned}
(i) \quad & \sum_{k \in \mathbf{Z}} \frac{\langle (: t(w)u(w) :) \circ_k v(w) \rangle}{(z-w)^{k+1}} \\
&= \sum_{n, m \geq 0} \frac{\langle : \partial^m t(w) u(w) \circ_n v(w) : \rangle \pm \langle : \partial^m u(w) t(w) \circ_n v(w) : \rangle}{m!(z-w)^{n-m+1}} \\
&\quad \pm \sum_{n, m \geq 0} \frac{\langle u(w) \circ_n (t(w) \circ_m v(w)) \rangle}{(z-w)^{n+m+2}} \\
&\quad + \sum_{m \geq 0} \frac{\langle : \partial^m (t(w)u(w)) v(w) : \rangle}{m!(z-w)^{-m}} \\
(ii) \quad & \pm \sum_{k \in \mathbf{Z}} \frac{\langle t(w) \circ_k : u(w)v(w) : \rangle}{(z-w)^{k+1}} \\
&= \sum_{n, m \geq 0} (-1)^{n+1} \frac{\langle u(w) \circ_n (t(w) \circ_m v(w)) \rangle}{(z-w)^{n+m+2}} \\
&\quad + \sum_{n, m \geq 0} (-1)^{n+1} \frac{\langle : \partial^m (u(w) \circ_n t(w)) v(w) : \rangle}{m!(z-w)^{n-m+1}}
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
& + \sum_{n \geq 0} \frac{\langle : u(w) t(w) \circ_n v(w) : \rangle}{(z-w)^{n+1}} \\
& + \sum_{m \geq 0} \frac{\langle : u(w) (\partial^m t(w)) v(w) : \rangle}{m! (z-w)^{-m}}
\end{aligned} \tag{3.20}$$

Proof: To prove (i), consider matrix coefficients on both sides of eqn. (3.17). By assumption of commutativity these matrix coefficients represent rational functions. Expanding both sides using Lemma 2.6, we get the first eqn. (i). The eqn. (ii) is derived similarly from (3.18). \square

By reading off coefficients of the $(z-w)^i$, we can use this lemma to simultaneously compute all products $: t(w)u(w) : \circ_k v(w)$, and $t(w) \circ_k : u(w)v(w) :$ in terms of the products among the constituents $t(w), u(w), v(w)$. Thus it is a kind of recursion relation for the products. In the examples below, we will see how it allows us to understand the structure of commutative QOAs.

Lemma 3.3 *If $t(z)^\pm u(w)^\pm = (-1)^{|t||u|} u(w)^\pm t(z)^\pm$, then $: t(z)u(w)v(x) := (-1)^{|t||u|} : u(w)t(z)v(x) :$.*

Proof: Applying the definition of the Wick product (and surpressing z, w, x):

$$\begin{aligned}
& : tuv : - (-1)^{|t||u|} : utv : \\
& = t^-(u^-v + (-1)^{|u||v|}vu^+) + (-1)^{|t|(|u|+|v|)}(u^-v + (-1)^{|u||v|}vu^+)t^+ \\
& - (-1)^{|t||u|} \left(u^-(t^-v + (-1)^{|t||v|}vt^+) + (-1)^{|u|(|t|+|v|)}(t^-v + (-1)^{|t||v|}vt^+)u^+ \right) \\
& = 0. \quad \square
\end{aligned} \tag{3.21}$$

3.1 Examples

Let $QO(V)^- = \{u(z)^- \mid u(z) \in QO(V)\}$. This space is obviously closed under differentiation and the Wick product. It follows that the space is also closed under all \circ_n , n negative. Also observe that for any $u(z), v(z) \in QO(V)$, we have $u(z)^-v(w)^- =: u(z)^-v(w)^- :. It follows that the products \circ_n , $n = 0, 1, \dots$, restricted to $QO(V)^-$, all vanish. Thus $QO(V)^-$ is a local QOA.$

Let $LO(V)$ be the algebra of graded linear operators on V . We can regard each operator A as a formal series with just the constant term. This makes $LO(V)$ a subspace of $QO(V)$. It is obvious that every \circ_n restricted to $LO(V)$ vanishes except for $n = -1$, in which case \circ_{-1} is the usual product on $LO(V)$. Thus $LO(V)$ is a very degenerate example of a QOA. Obviously, any commutative subalgebra of $LO(V)$ is a commutative QOA.

Let \mathcal{C} be the Clifford algebra with the generators $b(n), c(n)$ ($n \in \mathbf{Z}$) and the relations [11][16][1]

$$\begin{aligned}
b(n)c(m) + c(m)b(n) &= \delta_{n,-m-1} \\
b(n)b(m) + b(m)b(n) &= 0 \\
c(n)c(m) + c(m)c(n) &= 0
\end{aligned} \tag{3.22}$$

Let λ be a fixed integer. The algebra \mathcal{C} becomes \mathbf{Z} -bigraded if we define the degrees $|b(n)| = -|c(n)| = -1$, $||b(n)|| = \lambda - n - 1$, $||c(n)|| = -\lambda - n$. Let Λ^* be the graded irreducible \mathcal{C}^* -module with generator $\mathbf{1}$ and relations

$$b(m)\mathbf{1} = c(m)\mathbf{1} = 0, \quad m \geq 0 \quad (3.23)$$

Let $b(z), c(z)$ be the quantum operators

$$\begin{aligned} b(z) &= \sum_m b(m)z^{-m-1} \\ c(z) &= \sum_m c(m)z^{-m-1} \end{aligned} \quad (3.24)$$

Let $O(b, c)$ be the smallest QOA containing $b(z), c(z)$.

Proposition 3.4 *The QOA $O(b, c)$ is commutative. It has a basis consisting of the monomials*

$$: \partial^{n_1} b(z) \cdots \partial^{n_i} b(z) \partial^{m_1} c(z) \cdots \partial^{m_j} c(z) : \quad (3.25)$$

with $n_1 > \dots > n_i \geq 0$, $m_1 > \dots > m_j \geq 0$.

Proof: Computing the OPE of $b(z), c(w)$, we have

$$\begin{aligned} b(z)c(w) &= (z-w)^{-1} + : b(z)c(w) : \\ c(w)b(z) &= (w-z)^{-1} + : c(w)b(z) : \\ : b(z)c(w) : &= - : c(w)b(z) : . \end{aligned} \quad (3.26)$$

It follows that $b(z)$ and $c(z)$ commute. Also $b(z), c(z)$ each commutes with itself, hence they form a pairwise commuting set. By Lemma 3.1, they generate a commutative QOA.

If each $u_1(z), \dots, u_k(z)$ is of the form $\partial^n b(z)$ or $\partial^m c(z)$, let's call $: u_1(z) \cdots u_k(z) :$ a monomial of degree k . We claim that it's proportional to some monomial (3.25) with $n_1 > \dots > n_i \geq 0$, $m_1 > \dots > m_j \geq 0$. If $t(z), u(z)$ each is of the form $\partial^n b(z)$ or $\partial^m c(z)$, it is easy to check that $t(z)^\pm u(z)^\pm = -u(z)^\pm t(z)^\pm$. It follows from Lemma 3.3 that $: t(z)u(z)v(z) := -u(z)t(z)v(z) :$ for any element $v(z) \in O(b, c)$. This shows that $: u_1(z) \cdots u_k(z) :$ is equal to $(-1)^\sigma : u_{\sigma(1)}(z) \cdots u_{\sigma(k)}(z) :$ for any permutation σ of $1, \dots, k$.

Let O' be the linear span of the monomials (3.25). We now show that $A \circ_k B \in O'$ for any k and any two monomials A, B , hence $O(b, c) = O'$. We will do a double induction on the degrees of A and B . Case 1: let $A = t(z)$, $B = : u(z)v(z) :$ with $t(z), u(z)$ each monomial of degree 1, and $v(z)$ of any degree. If $v(z) = 1$, then by (3.26) $t(w) \circ_k : u(w)v(w) : \in O'$. By induction on the degree of $v(z)$ and applying Lemma 3.2(ii), we see that $t(w) \circ_k : u(w)v(w) : \in O'$. This shows that $A \circ_k B \in O'$ for A of degree 1, B of any degree. Now case 2: suppose $A = : t(z)u(z) :$,

$B = v(z)$, where $t(z)$ is of degree 1 and $u(z), v(z)$ of any degree. By induction on the degree of $u(z)$, it's clear from Lemma 3.2(i) that this case reduces to case 1.

Finally we must show that the monomials (3.25) are linearly independent. Define a map $O(b, c) \rightarrow \Lambda$ by $u(z) \mapsto u(-1)\mathbf{1}$. We see that this map gives a 1-1 correspondence between the set of monomials (3.25) and a basis of Λ . This completes the proof. \square .

Let $M(\kappa, 0)$ be the Verma module of the Virasoro algebra with highest weight $(\kappa, 0)$ and vacuum vector v_0 . Let $M(\kappa)$ be the quotient of $M(\kappa, 0)$ by the submodule generated by $L_{-1}v_0$. Let $O_\kappa(L)$ be the QOA generated by $L(z) = \sum L_n z^{-n-2}$ in $QO(M(\kappa))$.

Proposition 3.5 (see [3][10]) *The QOA $O_\kappa(L)$ is commutative. It has a basis consisting of monomials*

$$: \partial^{n_1} L(z) \cdots \partial^{n_i} L(z) : \quad (3.27)$$

with $n_1 \geq \dots \geq n_i \geq 0$.

Proof: A direct computation gives

$$\begin{aligned} [L(z)^+, L(w)] &= \frac{\kappa}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1} \\ [L(z)^-, L(w)] &= -\frac{\kappa}{2}(w-z)^{-4} - 2L(w)(w-z)^{-2} + \partial L(w)(w-z)^{-1}. \end{aligned} \quad (3.28)$$

But we also have $L(z)L(w) = [L(z)^+, L(w)] + :L(z)L(w):$, and $L(w)L(z) = -[L(z)^-, L(w)] + :L(z)L(w):$. Combining these with (3.28), it is obvious that $\langle L(z)L(w) \rangle$ and $\langle L(w)L(z) \rangle$ represent the same rational function. Thus $L(z)$ commutes with itself as a quantum operator. By Lemma 3.1, $O_\kappa(L)$ is commutative.

Let O' be the linear span of the monomials (3.27) with $n_1, \dots, n_i \geq 0$ *unrestricted*. To show O' is closed under all the products (hence $O_\kappa(L) = O'$), we apply induction and Lemma 3.2 as in the case of $O(b, c)$ above. We now show that we can restrict to those monomials (3.27) with $n_1 \geq \dots \geq n_i \geq 0$, and that the resulting monomials form a basis. First by direct computation, we see that $O_\kappa(L)$ is a *Vir*-module defined by the action

$$L(n) \cdot u(z) = L(z) \circ_n u(z). \quad (3.29)$$

Since $O_\kappa(L)$ is spanned by the monomials (3.27), and because $L(-n-1) \cdot u(z) = \frac{1}{n!} : \partial^n L(z) u(z) :$ for $n \geq 0$, it follows that the module is cyclic. Thus we have a unique onto map of *Vir*-modules $M(\kappa) \rightarrow O_\kappa(L)$ sending v_0 to 1. But $M(\kappa)$ has a PBW basis consisting of $L(-n_1-1) \cdots L(-n_i-1)v_0$, $n_1 \geq \dots \geq n_i \geq 0$. This shows that the monomials (3.27) with $n_1 \geq \dots \geq n_i \geq 0$ span $O_\kappa(L)$. Now define a map $O_\kappa(L) \rightarrow M(\kappa)$ by $u(z) \mapsto u(-1)v_0$. This is the inverse to the previous map, hence it must map a basis to a basis. \square

4 BRST cohomology algebras

Definition 4.1 A conformal QOA with central charge κ is a pair (O, f) , where O is a commutative QOA equipped with a homomorphism $f : O_\kappa(L) \rightarrow O$ such that for every homogeneous $u(z) \in O$,

$$fL(z)u(w) = \cdots + ||u||u(w)(z-w)^{-2} + \partial u(w)(z-w)^{-1} + :fL(z)u(w): \quad (4.30)$$

where “ \cdots ” denotes the higher order polar terms. In other words, $fL(z) \circ_1 u(z) = ||u||u(w)$ and $fL(z) \circ_0 u(z) = \partial u(z)$. For simplicity, we sometimes write $f : O_\kappa(L) \rightarrow O$, or simply $O_\kappa(L) \rightarrow O$, to denote a conformal QOA. A homomorphism $(O, f) \xrightarrow{h} (O', f')$ of conformal QOAs is a homomorphism of QOAs $h : O \rightarrow O'$ such that $h \circ f = f'$.

Lemma 4.2 Let O be a commutative QOA generated by a set S . Let $X(z) \in O$ such that for all $u(z) \in S$,

$$\begin{aligned} X(z)u(w) &= \cdots + ||u||u(w)(z-w)^{-2} + \partial u(w)(z-w)^{-1} + :X(z)u(w): \\ X(z)X(w) &= \frac{\kappa}{2}(z-w)^{-4} + 2X(w)(z-w)^{-2} + \partial X(w)(z-w)^{-1} + :X(z)X(w): \end{aligned} \quad (4.31)$$

Then there's a unique homomorphism $f : O_\kappa(L) \rightarrow O$ such that $fL(z) = X(z)$. Moreover (O, f) is a conformal QOA with central charge κ .

Proof: Uniqueness of f is clear because $O_\kappa(L)$ is generated by $L(z)$.

Step 1: We claim that O is a Vir-module defined by the action:

$$L(n) \cdot u(z) = X(z) \circ_n u(z). \quad (4.32)$$

We must show that

$$L(m) \cdot L(n) \cdot u(z) - L(n) \cdot L(m) \cdot u(z) = (m-n)L(m+n-1) \cdot u(z) + \frac{\kappa}{12}m(m-1)(m-2)\delta_{n+m-2}u(z). \quad (4.33)$$

From the OPE of $X(z)$, we get (cf. (3.28)):

$$\begin{aligned} LHS &= Res_{z_1} Res_{z_2} [X(z_2), X(z_1)]u(z)(z_2-z)^m(z_1-z)^n \\ &\quad - Res_{z_1} Res_{z_2} u(z)[X(z_2), X(z_1)](-z+z_2)^m(-z+z_1)^n \\ &= Res_{z_1} Res_{z_2} (z_2-z)^m(z_1-z)^n \left(\frac{\kappa}{12}\partial_{z_1}^3 + 2X(z_1)\partial_{z_1} + \partial_{z_1}X(z_1) \right) \delta(z_1, z_2)u(z) \\ &\quad - Res_{z_1} Res_{z_2} u(z)(-z+z_2)^m(-z+z_1)^n \left(\frac{\kappa}{12}\partial_{z_1}^3 + 2X(z_1)\partial_{z_1} + \partial_{z_1}X(z_1) \right) \delta(z_1, z_2)u(z) \end{aligned} \quad (4.34)$$

where $\delta(z_1, z_2) = (z_2 - z_1)^{-1} - (-z_1 + z_2)^{-1}$. For any Laurent polynomial $g(z_1, z_2)$, and any formal series $h(z_1, z_2)$, we have the identities

$$\begin{aligned} g(z_1, z_2)\delta(z_1, z_2) &= g(z_1, z_1)\delta(z_1, z_2) \\ \text{Res}_{z_1}\delta(z_1, z_2) &= 1 \\ \text{Res}_{z_1}g(z_1, z_2)\partial_{z_1}^k h(z_1, z_2) &= (-1)^k \text{Res}_{z_1}(\partial_{z_1}^k g(z_1, z_2)) h(z_1, z_2). \end{aligned} \quad (4.35)$$

Applying these and continuing the above computation:

$$\begin{aligned} LHS &= \text{Res}_{z_1} \left(-\frac{\kappa}{12} n(n-1)(n-2)(z_1 - z)^{m+n-3} \right. \\ &\quad \left. - 2X(z_1)n(z_1 - z)^{m+n-1} - 2\partial_{z_1}X(z_1)(z_1 - z)^{m+n} + \partial_{z_1}X(z_1)(z_1 - z)^{m+n} \right) u(z) \\ &\quad - \text{Res}_{z_1} \left(-\frac{\kappa}{12} n(n-1)(n-2)(-z + z_1)^{m+n-3} \right. \\ &\quad \left. - 2X(z_1)n(-z + z_1)^{m+n-1} - 2\partial_{z_1}X(z_1)(-z + z_1)^{m+n} + \partial_{z_1}X(z_1)(-z + z_1)^{m+n} \right) u(z) \\ &= -\frac{\kappa}{12} n(n-1)(n-2) 1 \circ_{m+n-3} u(z) + (m-n)X(m+n-1) \cdot u(z). \end{aligned} \quad (4.36)$$

This proves (4.33). Thus we've a map $f : O_\kappa(L) \cong M(\kappa) \rightarrow O$ with $1 \mapsto 1$. Moreover, we have

$$f(: \partial^{n_1} L(z) \cdots \partial^{n_i} L(z) :) =: \partial^{n_1} X(z) \cdots \partial^{n_i} X(z) :. \quad (4.37)$$

Step 2: We'll show that f is a QOA homomorphism, ie. for all $A, B \in O_\kappa(L)$ and integers n ,

$$f(A \circ_n B) = fA \circ_n fB. \quad (4.38)$$

It's enough to do it for A, B being monomials $: u_1(z) \cdots u_k(z) :$ where each u has degree 1, ie. of the form $\partial^n L(z)$. Once again by double induction on the degrees of A, B , it's easy to show that $A \circ_n B$ can be reduced, by *Lemma 3.2* and the OPE $L(z)L(w)$ only, to a linear sum of the above monomials. By (4.37), $fA \circ_n fB$ must be reduced, by *Lemma 3.2* and the OPE $X(z)X(w)$ only, to a linear sum *identical* to the reduction of $f(A \circ_n B)$. This is so because $L(z)$ and $X(z)$ have identical OPEs. This proves (4.38) for all A, B .

Step 3: We must show that $fL(z) = X(z)$ has the desired properties: $X(z) \circ_0 u(z) = \partial u(z)$ and $X(z) \circ_1 u(z) = ||u||u(z)$, for all $u(z) \in O$. By assumption, they hold for all $u(z)$ in the generating set S . Checking the properties is an easy exercise applying the formulas $X(z) \circ_0 \partial u(z) = [X(0), u(z)]$, $X(z) \circ_1 u(z) = [X(1) - X(0)z, u(z)]$ and applying induction. \square

Consider, as an example, $O(b, c)$. For a fixed λ , let

$$X(z) = (1 - \lambda) : \partial b(z) c(z) - \lambda : b(z) \partial c(z) : \quad (4.39)$$

and $S = \{b(z), c(z)\}$. Then we have, by direct computation [11],

$$\begin{aligned} X(z)b(w) &= \lambda b(w)(z-w)^{-2} + \partial b(w)(z-w)^{-1} + : X(z)b(w) : \\ X(z)c(w) &= (1 - \lambda)c(w)(z-w)^{-2} + \partial c(w)(z-w)^{-1} + : X(z)c(w) : \\ X(z)X(w) &= \frac{\kappa}{2}(z-w)^{-4} + 2X(w)(z-w)^{-2} + \partial X(w)(z-w)^{-1} + : X(z)X(w) :. \end{aligned} \quad (4.40)$$

where $\kappa = -12\lambda^2 + 12\lambda - 2$. It follows that we have a homomorphism $f_\lambda : O_\kappa(L) \rightarrow O(b, c)$ such that $(O(b, c), f_\lambda)$ is a conformal QOA with central charge κ .

Similarly the pair $(O_\kappa(L), id)$ is itself a conformal QOA. Thus by definition, it is the initial object in the category of conformal QOAs with central charge κ .

4.1 The BRST construction

It is evident that if $(O, f), (O', f')$ are conformal QOAs on the respective spaces V, V' with central charges κ, κ' , then $(O \otimes O', f \otimes f')$ is a conformal QOA on $V \otimes V'$ with central charge $\kappa + \kappa'$. From now on we fix $\lambda = 2$ which means that $(O(b, c), f_\lambda)$ now has central charge -26. Let (O, f) be any conformal QOA with central charge κ and consider

$$C^*(O) = O(b, c) \otimes O \quad (4.41)$$

where $*$ denotes the total first degree. For simplicity, we write $L^C(z) = f_\lambda L(z) + fL(z)$.

Proposition 4.3 *For every conformal QOA O , there is a unique homogeneous element $J_O(z) \in C^*(O)$ with the following properties:*

- (i) (Cartan identity) $J_O(z)b(w) = L^C(w)(z - w)^{-1} + : J_O(z)b(w) :$
- (ii) (Universality) If $(O, f) \rightarrow (O', f')$ is a homomorphism of conformal QOAs, then the induced homomorphism $C^*(O) \rightarrow C^*(O')$ sends $J_O(z)$ to $J_{O'}(z)$.

Proof: Since the category of conformal QOAs with central charge κ has $(O_\kappa(L), id)$ as the initial object, if we can show that there is a unique $J_{O_\kappa(L)}$ satisfying property (i), then (ii) implies that the same holds for every other object in that category.

Property (i) implies $|J_{O_\kappa(L)}| = 1 = ||J_{O_\kappa(L)}||$. Let's list a basis of $C^1(O_\kappa(L))[1]$ given by Propositions 3.4, 3.5: $: c(z)L(z) :, : b(z)c(z)\partial c(z) :, \partial^2 c(z)$. Take a linear combination of these elements and compute its OPE with $b(z)$. Now requiring property (i), we determine the coefficients of the linear combination and get

$$J_{O_\kappa(L)}(z) = : c(z)L(z) : + : b(z)c(z)\partial c(z) : . \quad (4.42)$$

Now given a conformal QOA (O, f) , the induced map $f^* : C^*(O_\kappa(L)) \rightarrow C^*(O)$ sends $J_{O_\kappa(L)}(z)$ to $J_O(z) = : c(z)fL(z) : + : b(z)c(z)\partial c(z) :$. This completes our proof. \square .

It follows from property (i) that

$$L^C(z) = J_O(z) \circ_0 b(z) = [Q, b(z)] \quad (4.43)$$

where $Q = Res_z J_O(z)$.

Lemma 4.4 [17][6][7] *Let (O, f) be a conformal QOA with central charge κ . Then $Q^2 = 0$ iff $\kappa = 26$.*

Proof: We'll drop the subscript for J_O and write $fL(z)$ as $L(z)$. Let's compute $2Q^2 = [Q, Q] = \text{Res}_w[Q, J(w)] = \text{Res}_w J(w) \circ_0 J(w)$. Since $J(w) \circ_0 J(w)$ is the coefficient of $(z - w)^{-1}$ in the OPE $J(z)J(w)$, we can extract this term from the OPE. Now $J(z)J(w)$ is the sum of 4 terms:

$$\begin{aligned} (i) \quad & c(z)L(z)c(w)L(w) \\ (ii) \quad & c(z)L(z) : b(w)c(w)\partial c(w) : \\ (iii) \quad & : b(z)c(z)\partial c(z) : c(w)L(w) \\ (iv) \quad & : b(z)c(z)\partial c(z) :: b(w)c(w)\partial c(w) : . \end{aligned} \tag{4.44}$$

Extracting the coefficient of $(z - w)^{-1}$ (which is done by applying Lemma 3.2 repeatedly) in each of these 4 OPEs, we get respectively (surpressing w):

$$\begin{aligned} (i) \quad & 2\partial c \, cL + \frac{\kappa}{12}\partial^3 c \, c \\ (ii) \quad & c\partial c \, L \\ (iii) \quad & c\partial c \, L \\ (iv) \quad & \frac{3}{2}\partial(\partial^2 c \, c) - \frac{13}{6}\partial^3 c \, c \end{aligned} \tag{4.45}$$

Thus $J(w) \circ_0 J(w) = \frac{3}{2}\partial(\partial^2 c(w) \, c(w)) + \frac{\kappa-26}{12}\partial^3 c(w) \, c(w)$. The Res_w of this is zero iff $\kappa = 26$. \square

Recall that (Lemma 2.2) $[Q, -] = J(z) \circ_0$ is a derivation of the QOA $C^*(O)$. For $\kappa = 26$, which we assume from now on, $[Q, -]$ becomes a differential on $C^*(O)$ and we have a cochain complex

$$[Q, -] : C^*(O) \longrightarrow C^{*+1}(O). \tag{4.46}$$

It is called *the BRST complex* associated to O . Its cohomology will be denoted as $H^*(O)$. All the operations \circ_n on $C^*(O)$ descend to the cohomology. However, all but one is trivial.

Theorem 4.5 [32][34][26] *The Wick product \circ_{-1} induces a graded commutative associative product on $H^*(O)$ with unit element represented by the identity operator. Moreover, every cohomology class is represented by a quantum operator $u(z)$ with $\|u\| = 0$.*

Proof: Let $u(z), v(z)$ be two elements of $C^*(O)$ annihilated by $[Q, -]$. Consider the rational functions represented by $\langle u(z)v(w) \rangle, \pm \langle v(w)u(z) \rangle$. Expand the second one in the region $|w| > |z - w|$:

$$\langle v(w)u(z) \rangle = \sum_{m \geq 0} \langle v(z) \circ_m u(z) \rangle (w - z)^{-m-1} + \langle : v(w)u(z) : \rangle$$

$$\begin{aligned}
&= \sum_{m \geq 0} \sum_{k \geq 0} \partial^k (\langle v(w) \circ_m u(w) \rangle) \frac{(-1)^k}{k!} (w - z)^{k-m-1} \\
&\quad + \sum_{k \geq 0} \langle : v(w) \partial^k u(w) : \rangle \frac{(-1)^k}{k!} (w - z)^k.
\end{aligned} \tag{4.47}$$

By commutativity, this coincides with the following rational function times $(-1)^{|v||u|}$:

$$\langle u(z)v(w) \rangle = \sum_{n \geq 0} \langle u(w) \circ_n v(w) \rangle (z - w)^{-n-1} + \langle : u(z)v(w) : \rangle. \tag{4.48}$$

Expanding this in the same region $|w| > |z - w|$, we see that its coefficient of $(z - w)^0$ is $\langle : u(w)v(w) : \rangle$. Equating this with the same coefficient in (4.47) (with appropriate sign), we get

$$\sum_{m \geq 0} \frac{\partial^{m+1} (\langle v(w) \circ_m u(w) \rangle)}{(m+1)!} + \langle : v(w)u(w) : \rangle = (-1)^{|v||u|} \langle : u(w)v(w) : \rangle. \tag{4.49}$$

We can write the first term on the left hand side as $\langle \partial A(w) \rangle$ where $A(w)$ is a quantum operator. (We can do this because the sum is finite by locality.) Note that because $[Q, -]$ is a derivation on the products \circ_m and because $[Q, -]$ annihilates all $\partial^i u(w), \partial^j v(w)$, it follows that $[Q, A(w)] = 0$. Thus we have

$$\begin{aligned}
\langle : v(w)u(w) : \rangle - (-1)^{|v||u|} \langle : u(w)v(w) : \rangle &= \langle \partial A(w) \rangle \\
&= \langle L^C(w) \circ_0 A(w) \rangle \\
&= \langle [Q, b(w)] \circ_0 A(w) \rangle \\
&= \langle [Q, b(w) \circ_0 A(w)] \rangle.
\end{aligned} \tag{4.50}$$

This implies that $: v(w)u(w) :$, $(-1)^{|v||u|} : u(w)v(w) :$ are cohomologous to each other.

To prove associativity, let $u(z), v(z), t(z)$ represent three cohomology classes. We will compute $\langle : u(z)v(z) : t(w) \rangle$ in two different ways. First we have

$$\begin{aligned}
\langle : u(z)v(z) : t(w) \rangle &= \langle (u(z)^- v(z) + (-1)^{|u||v|} v(z) u(z)^+) t(w) \rangle \\
&= \sum_{n \geq 0} \langle : u(z)v(w) \circ_n t(w) : \rangle (z - w)^{-n-1} + \langle : u(z)v(z)t(w) : \rangle \\
&\quad + (-1)^{|u||v|} \sum_{m \geq 0} \langle v(z)u(w) \circ_m t(w) \rangle (z - w)^{-m-1}.
\end{aligned} \tag{4.51}$$

The right hand side is now a rational function which we can expand in the region $|w| > |z - w|$ and extract the coefficient of $(z - w)^0$. On the other hand this coefficient must be equal to $\langle : (: u(w)v(w) :) t(w) : \rangle$. Thus we get

$$\begin{aligned}
\langle : (: u(w)v(w) :) t(w) : \rangle &= \langle : u(w) (: v(w)t(w) :) : \rangle \\
&\quad + \sum_{n \geq 0} \frac{\langle : \partial^{n+1} u(w) v(w) \circ_n t(w) : \rangle}{(n+1)!} \\
&\quad + (-1)^{|u||v|} \sum_{m \geq 0} \frac{\langle : \partial^{m+1} v(w) u(w) \circ_m t(w) : \rangle}{(m+1)!}.
\end{aligned} \tag{4.52}$$

Using the fact that $\partial A(w) = [Q, b(w)] \circ_0 A(w)$ for any $A(w)$, and the fact that $[Q, -]$ annihilates $u(w), v(w), t(w)$, we see that $: (: u(w)v(w) :) t(w) :$ and $: u(w) (: v(w)t(w) :) :$ are cohomologous to each other.

Checking that the quantum operator 1 represents the unit is a trivial exercise. Finally we want to show that an element $u(z)$, with $||u|| \neq 0$, annihilated by $[Q, -]$ is cohomologous to zero. Recall that $L^C(z) \circ_1 u(z) = ||u||u(z)$. But the left hand side is $[Q, b(z) \circ_1 u(z)]$ which is cohomologous to zero. \square

5 Batalin-Vilkovisky Algebras

Let A^* be a \mathbf{Z} graded commutative associative algebra. For every $a \in A$, let l_a denote the linear map on A given by the left multiplication by a . Recall that a (graded) derivation d on A is a homogeneous linear operator such that $[d, l_a] - l_{da} = 0$ for all a . A BV operator [33][31][14] Δ on A^* is a linear operator of degree -1 such that:

- (i) $\Delta^2 = 0$;
- (ii) $[\Delta, l_a] - l_{\Delta a}$ is a derivation on A for all a , ie. Δ is a *second* order derivation.

A BV algebra is a pair (A, Δ) where A is a graded commutative algebra and Δ is a BV operator on A . The following is an elementary but fundamental lemma:

Lemma 5.1 [20][14][29] *Given a BV algebra (A, Δ) , define the BV bracket $\{, \}$ on A by:*

$$(-1)^{|a|} \{a, b\} = [\Delta, l_a]b - l_{\Delta a}b.$$

Then $\{, \}$ is a graded Lie bracket on A of degree -1.

By property (ii) above, it follows immediately that for every $a \in A$, $\{a, -\}$ is a derivation on A . Thus a BV algebra is a special case of an odd Poisson algebra which, in mathematics, is also known as a Gerstenhaber algebra [12][13]. We note that A^1 is canonically a Lie algebra.

Consider now the linear operator $\Delta : C^*(O) \longrightarrow C^{*-1}(O)$, $u(z) \mapsto b(z) \circ_1 u(z)$.

Theorem 5.2 [26] *The operator Δ descends to the cohomology $H^*(O)$. Moreover, it is a BV operator on the commutative algebra $H^*(O)$. Thus $H^*(O)$ is naturally a BV algebra.*

Proof: By the theorem above, it is enough consider the action of Δ on elements $u(z)$ with $||u|| = 0$. If $[Q, u(z)] = 0$, we have

$$[Q, \Delta u(z)] = [Q, b(z)] \circ_1 u(z) = L^C(z) \circ_1 u(z) = ||u(z)||u(z) = 0. \quad (5.53)$$

Thus Δ is well-defined on the cohomology.

Also we have

$$\Delta^2 u(z) = [b(1) - b(0)z, [b(1) - b(0)z, u(z)]] = 0 \quad (5.54)$$

because the b 's anticommute.

We define the following bilinear operation:

$$(-1)^{|u|} \{u(z), v(z)\} = \Delta(: u(z)v(z) :) - : (\Delta u(z))v(z) : - (-1)^{|u|} : u(z)(\Delta v(z)) :. \quad (5.55)$$

We claim that the following identity holds:

$$\{u(z), : v(z)t(z) : \} = : \{u(z), v(z)\}t(z) : + (-1)^{(|u|-1)|v|} : v(z)\{u(z), t(z)\} : \quad (5.56)$$

ie. the bracket is a derivation in the second argument. This says that Δ is a second order derivation, and hence proves that it is a BV operator on the cohomology.

Lemma 5.3 *For any homogeneous quantum operators $A(z), B(z), C(z)$, the following holds:*

$$\begin{aligned} A(z) \circ_1 (: B(z)C(z) :) &= : (A(z) \circ_1 B(z))C(z) : - (-1)^{|A||B|} : B(z)(A(z) \circ_1 C(z)) : \\ &= (A(z) \circ_0 B(z)) \circ_0 C(z). \end{aligned} \quad (5.57)$$

When $A(z) = b(z)$, the LHS of (5.57) is nothing but $(-1)^{|B|} \{B(z), C(z)\}$ while the RHS is clearly a derivation in the argument $C(z)$. Thus the lemma implies the identity (5.56).

Proof of lemma: The LHS of (5.57) is:

$$\begin{aligned} LHS &= [A(1) - A(0)z, : B(z)C(z) :] - : [A(1) - A(0)z, B(z)]C(z) : \\ &\quad - (-1)^{|A||B|} : B(z)[A(1) - A(0)z, C(z)] : \\ &= -[A(0)z, B(z)^-]C(z) - (-1)^{|B||C|+|A||C|}C(z)[A(0)z, B(z)^+] \\ &\quad - (-1)^{|A||B|} : B(z)[A(0)z, C(z)] : \\ &\quad + [A(0)z, B(z)]^- C(z) + (-1)^{(|A|+|B|)|C|}C(z)[A(0)z, B(z)]^+ \\ &\quad + (-1)^{|A||B|} : B(z)[A(0)z, C(z)] : \\ &= -[A(0)z, B(z)^-]C(z) - (-1)^{|B||C|+|A||C|}C(z)[A(0)z, B(z)^+] \\ &\quad + ([A(0), B(0)] + [A(0)z, B(z)^-])C(z) \\ &\quad + (-1)^{(|A|+|B|)|C|}C(z)([A(0)z, B(z)^+] - [A(0), B(0)]) \\ &= [A(0), B(0)]C(z) - (-1)^{(|A|+|B|)|C|}C(z)[A(0), B(0)] \\ &= [A(0), [B(0), C(z)]] \\ &= RHS. \end{aligned} \quad (5.58)$$

This proves our lemma and completes our proof of the theorem. \square

The two main theorems above were originally proved in [26] in the context of vertex operator algebras. (For related versions of Theorem 5.2, see [14][30][18][15].)

References

- [1] F. Akman, “The semi-infinite Weil complex of a graded Lie algebra”, Yale Thesis 1993.
- [2] P. Bouwknegt, K. Pilch, “The BV-algebra structure of W_3 cohomology”, preprint 1994, to appear in the Proceedings of “Gürsey Memorial Conference I: Strings and Symmetries”, eds. M. Serdaroglu et al.
- [3] A. Belavin, A.M. Polyakov and A.A. Zamolodchikov, “Infinite conformal symmetry in two dimensional quantum field theory”, Nucl. Phys. B241 (1984) 333.
- [4] R.E. Borcherds, “Vertex operator algebras, Kac-Moody algebras and the Monster”, Proc. Natl. Acad. Sci. USA. 83 (1986) 3068.
- [5] C.-Y. Dong and J. Lepowsky, “Generalized vertex algebras and relative vertex operators”, Prog. in Math. vol 112, Birkhauser, Boston, 1993.
- [6] B. Feigin, “Semi-infinite homology for Virasoro and Kac-Moody algebras”, Usp. Mat. Nauk. 39 (1984) 195-196.
- [7] I.B. Frenkel, H. Garland and G.J. Zuckerman, “Semi-infinite cohomology and string theory”, Proc. Nat. Acad. Sci. U.S.A. 83 (1986) 8442.
- [8] I.B. Frenkel, J. Lepowsky and A. Meurman, “Vertex Operator Algebras and the Monster”, Academic Press, New York, 1988.
- [9] I.B. Frenkel, Y. Huang and J. Lepowsky, “On axiomatic approaches to vertex operator algebras and modules”, Yale-Rutgers preprint, Dept. of Math., 1989.
- [10] I.B. Frenkel and Y. Zhu, “Vertex Operator Algebras associated to affine Lie algebras and the Virasoro algebra”, Duke Math J. 66 (1992) 123.
- [11] D. Friedan, E. Martinec and S. Shenker, “Conformal invariance, supersymmetry and string theory”, Nucl. Phys. B271 (1986) 93.
- [12] M. Gerstenhaber, “The cohomology structure of an associative ring”, Ann. of Math. 78, No.2 (1962) 267.
- [13] M. Gerstenhaber, “On the deformation of rings and algebras”, Ann. of Math. 79, No.1 (1964) 59.

- [14] E. Getzler, “Batalin-Vilkovisky algebras and two dimensional topological field theory”, preprint hep-th/9212043.
- [15] Y-Z. Huang, “Operadic formulation of topological vertex algebras and Gerstenhaber or Batalin-Vilkovisky algebras”, Univ of Penn preprint (1993).
- [16] V. Kac and A. Raina, *Highest weight representations of infinite dimensional Lie algebras*, Advanced Series in Math Physics, Vol. 2, World Scientific (1987).
- [17] M. Kato and K. Ogawa, “Covariant quantization of string based on BRS invariance”, Nucl. Phys. B212 (1983) 443.
- [18] T. Kimura, J. Stasheff and A. Voronov, “On operad structures of moduli spaces and string theory”, hep-th/9307114.
- [19] Y. Kosmann-Schwarzbach, “Exact Gerstenhaber algebras and Lie bialgebroids”, U.R.A. 169 CNRS preprint 1994.
- [20] J.-L. Koszul, “Crochet de Schouten-Nijenhuis et cohomologie”, Asterique, hors serie, (1985) 257-271.
- [21] H.-S. Li, “Local systems of vertex operators, vertex superalgebras and modules”, hep-th/9406185.
- [22] B.H. Lian and G.J. Zuckerman, “Some classical and quantum algebras”, hep-th/9404010.
- [23] B.H. Lian and G.J. Zuckerman, “New selection rules and physical states in 2d gravity; conformal gauge”, Phys. Lett B, 254, No.3,4 (1991) 417.
- [24] B.H. Lian and G.J. Zuckerman, “2d gravity with c=1 matter”, Phys. Lett. B266 (1991) 21.
- [25] B.H. Lian and G.J. Zuckerman, “Semi-infinite homology and 2D gravity (I)”, Commun. Math. Phys. 145 (1992) 561.
- [26] B.H. Lian and G.J. Zuckerman, “New perspectives on the BRST-algebraic structure in string theory”, Commun. Math. Phys. 154 (1993) 613.
- [27] G. Moore, “Finite in All Directions”, hep-th/9305139.
- [28] G. Moore and N. Seiberg, “Classical and quantum conformal field theory”, Comm. Math. Phys. 123 (1989) 177-254.
- [29] M. Penkava, “A note on BV algebras”, UC Davis preprint November 92.

- [30] M. Penkava and A. Schwarz, “On some algebraic structures arising in string theory”, UC Davis preprint hep-th/9212072.
- [31] A. Schwarz, “Geometry of Batalin-Vilkovisky quantization”, UC Davis preprint December 92.
- [32] E. Witten, “Ground ring of the two dimensional string theory”, Nucl. Phys. B373 (1992) 187.
- [33] E. Witten, “The anti-bracket formalism”, preprint IASSNS-HEP-90/9.
- [34] E. Witten and B. Zwiebach, “Algebraic structures and differential geometry in 2d string theory”, Nucl. Phys. B377 (1992) 55.

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY CAMBRIDGE, MA 02138. lian@math.harvard.edu

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY NEW HAVEN, CT 06520. gregg@math.yale.edu